# A Pointed Completion of the Freiman Category Restoring Finite (Co)limits and Translation Symmetry

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**Abstract**

The Freiman categories $\mathrm{FR}_k$ and $\mathrm{FR}_k^0$ organize additive sets and Freiman $k$-homomorphisms into a categorical framework. The normalized category $\mathrm{FR}_k^0$ is complete and cocomplete but excludes translations—morphisms central to additive combinatorics—because translations do not preserve the identity. Meanwhile $\mathrm{FR}_k$ admits translations but fails to have basic limits and colimits: equalizers and pullbacks may be empty, and coequalizers and pushouts can fail due to subgroup-closure issues. We construct an explicit "pointed completion" $\widehat{\mathrm{FR}}_k$ of $\mathrm{FR}_k$ by adjoining a distinguished basepoint element to every object and requiring morphisms to preserve this basepoint (without forcing it to be the ambient identity). This yields a finitely complete and finitely cocomplete category in which translations extend to isomorphisms, and in which $\mathrm{FR}_k^0$ embeds reflectively as the full subcategory of objects whose basepoint is 0. We also formulate a universal property: $\widehat{\mathrm{FR}}_k$ is the minimal finite-(co)limit setting in which the weak initial behavior of singletons in $\mathrm{FR}_k$ becomes genuinely initial. The construction provides an infrastructure layer for a modern categorical additive combinatorics program, enabling spans and cospans, diagrammatic gluing, and descent-type arguments without sacrificing translation symmetry.

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# 1 1. Motivation and overview: why $\mathrm{FR}_k$ lacks limits and colimits, why $\mathrm{FR}_k^0$ loses translations, and what a completion should accomplish.

The Freiman category $\mathrm{FR}_k$ is meant to isolate the combinatorial content of a finite additive set $A \subseteq G$ up to the relations detected by $k$-term additive configurations. Concretely, a Freiman $k$-homomorphism is only required to respect those additive equalities

$$a_1 + \cdots + a_k = a_1' + \cdots + a_k'$$

which hold inside the ambient group, and this looseness is precisely what makes $\mathrm{FR}_k$ convenient for additive-combinatorial arguments. At the same time, this looseness has a categorical cost: many universal constructions that are routine for algebraic categories are not available without additional structure. Our purpose is to explain why $\mathrm{FR}_k$ is not well behaved with respect to finite limits and colimits, why the usual "normalization" trick repairs some constructions but destroys translation symmetry, and why a pointed completion is the appropriate remedy.

A first obstruction already appears at the level of equalizers. Given parallel morphisms $f, g : (A, G) \to (B, H)$ in $\mathrm{FR}_k$, the set-theoretic equalizer

$$E := \{a \in A : f(a) = g(a)\}$$

need not be nonempty. In a category of *all* sets one simply allows $E = \varnothing$, but by definition objects of $\mathrm{FR}_k$ are *finite nonempty* subsets of abelian groups. Hence $(E, G)$ may fail to be an object. This is not a minor technicality: once equalizers fail to exist, finite limits fail in general, and many familiar constructions (kernels, pullbacks along diagonals, and so forth) are no longer guaranteed.

Dually, coequalizers suffer from an analogous nonexistence phenomenon. If $f, g : (A, G) \to (B, H)$ are parallel, a would-be coequalizer should identify points of $B$ forced equal by the relation generated from $f(a) \sim g(a)$. In a purely set-theoretic setting one forms a quotient set $B/\sim$. In $\mathrm{FR}_k$ we must additionally realize the result as a finite nonempty subset of some abelian group and ensure that the quotient map is a Freiman $k$-homomorphism. Even when a quotient set can be formed, there is generally no canonical way to choose an ambient group and embedding that makes the coequalizer universal among Freiman $k$-homomorphisms. Put differently, $\mathrm{FR}_k$ does not come with a built-in notion of "Freiman quotient" which is stable under the required universal property.

A separate, more conceptual obstruction concerns coproducts and pushouts. In $\mathrm{FR}_k$ the object $(A, G)$ is invariant under translations in the ambient group: for any $g \in G$, translation $\tau_g : A \to A + g$ is a Freiman $k$-isomorphism. Consequently, $A$ has no distinguished "origin" or "zero element" intrinsic to its isomorphism class. However, colimit constructions typically require choosing how to place multiple pieces relative to each other. For instance, to model a coproduct of $(A, G)$ and $(B, H)$, one would like a disjoint union in some ambient group so that there are canonical inclusion morphisms. If we attempt to realize such a disjoint union inside a single abelian group by translating one component far away from the other, then different translation choices are a priori different objects of $\mathrm{FR}_k$, and there is no canonical choice. Moreover, Freiman morphisms are sensitive to additive relations, so placing two components too close can create unintended $k$-term relations spanning both components, which obstructs universality. Thus, even if coproduct-like models can be engineered case-by-case, $\mathrm{FR}_k$ lacks a functorial construction compatible with the universal property.

One might try to resolve this by imposing a normalization condition, thereby passing to the category $\mathrm{FR}_k^0$ of additive sets containing 0 and Freiman morphisms preserving 0. In that setting, a copy of $A$ is always positioned so that its "distinguished" element is 0, and this rigidifies certain colimit constructions: when gluing along maps that preserve 0, there is at least a fixed point at which identifications may occur. However, the normalization comes at the expense of an essential symmetry. Translating $A$ inside $G$ is no longer a morphism unless the translation fixes 0, i.e. unless it is trivial. From the perspective of additive combinatorics, this loss is significant: many natural operations are affine rather than linear, and arguments frequently treat $A$ and $A+g$ as indistinguishable. The category $\mathrm{FR}_k^0$ forces us to choose a specific translate and then forbids moving away from it.

We therefore seek a completion which simultaneously addresses the two issues: it should enforce nonemptiness in limit/colimit constructions (so that equalizers, pullbacks, coequalizers, and pushouts exist), and it should preserve translation symmetry in a controlled way. The guiding analogy is the passage from sets to pointed sets. In pointed sets, one adjoins a basepoint so that constructions that would otherwise be empty can be represented by the basepoint alone, and coproducts become wedge sums formed by identifying basepoints. The basepoint does not erase information; rather, it supplies a canonical element that can absorb degeneracies (such as an empty equalizer) while remaining functorial.

In our Freiman setting, the basepoint plays two intertwined roles. First, it serves as a canonical "fallback" element, ensuring that when a limit would otherwise be empty (e.g. an equalizer), we still obtain a nonempty object by taking the basepoint alone. Second, it provides a distinguished point with respect to which we can glue objects in colimits without committing to a rigid ambient origin. Intuitively, instead of forcing each additive set to contain 0 inside its ambient group, we adjoin a new formal point which is declared to be preserved by morphisms. This basepoint behaves like the unique element of a singleton object, but unlike normalization it does not pick out a translate inside $G$ and hence does not destroy the ability to translate the rest of the set.

This approach also clarifies why we insist on $k \geq 2$. When $k = 1$, every set map is a Freiman 1-homomorphism, so the category forgets essentially all additive structure; the distinction between "structured" and "unstructured" embeddings collapses, and categorical pathologies become artifacts of having chosen a weak notion of morphism. For $k \geq 2$, additive relations are genuinely constrained, and the presence of a basepoint-preservation condition becomes meaningful: it enforces a minimal rigidity needed to make universal constructions canonical while still permitting the affine symmetries that are natural for additive sets.

The completion we develop can thus be viewed as the minimal modification of $\mathrm{FR}_k$ that achieves finite completeness and cocompleteness without abandoning translation invariance. We do not aim to impose a full ambient-group homomorphism structure (which would be too rigid for many combinatorial applications). Rather, we keep Freiman $k$-homomorphisms as morphisms, but we enrich objects by a marked point and require morphisms to respect it. With this small change, the missing limits and colimits can be constructed in a manner parallel to pointed sets, and translations remain available because they can be implemented as basepoint-preserving isomorphisms after a suitable canonical pointing. In the subsequent development we make these claims precise by defining the based category and exhibiting explicit constructions of finite limits and colimits within it.

## 2 2. The category $\widehat{\mathrm{FR}}_k$ of based additive sets: objects $(A, G, a_0)$, basepoint-preserving Freiman morphisms, and the role of basepoints in preventing emptiness.

We introduce a pointed variant of the Freiman category in which each additive set carries a distinguished element. Fix $k \geq 2$. An object of the based Freiman category $\widehat{\mathrm{FR}}_k$ is a triple

$$(A, G, a_0),$$

where $G$ is an abelian group, $A \subseteq G$ is a finite nonempty subset, and $a_0 \in A$ is a chosen basepoint. A morphism

$$f : (A, G, a_0) \longrightarrow (B, H, b_0)$$

is a Freiman $k$-homomorphism $f : A \to B$ (in the usual sense) such that $f(a_0) = b_0$. Thus the additional structure is minimal: we do not impose that $a_0 = 0$ in the ambient group, nor do we require morphisms to arise from ambient group homomorphisms; we merely insist that the marked point be preserved.

Since the Freiman $k$-condition will be used repeatedly, we record the convention. A map $f : A \to B$ between subsets of abelian groups is a Freiman $k$-homomorphism if for every choice of $a_1, \dots, a_k, a_1', \dots, a_k' \in A$ satisfying

$$a_1 + \cdots + a_k = a_1' + \cdots + a_k' \qquad \text{in } G,$$

we have

$$f(a_1) + \cdots + f(a_k) = f(a_1') + \cdots + f(a_k') \qquad \text{in } H.$$

Identities and compositions of basepoint-preserving Freiman $k$-homomorphisms are again basepoint-preserving Freiman $k$-homomorphisms, so $\widehat{\mathrm{FR}}_k$ is a well-defined category. There is also a forgetful functor $\widehat{\mathrm{FR}}_k \to \mathrm{FR}_k$ discarding the basepoint; we emphasize, however, that the based category is not merely a rephrasing of $\mathrm{FR}_k$: the basepoint constraint changes which diagrams admit universal cones.

The principal reason for passing to $\widehat{\mathrm{FR}}_k$ is that the distinguished point prevents the "emptiness" pathologies that obstruct finite limits in $\mathrm{FR}_k$. The equalizer example becomes completely transparent. Suppose we have parallel morphisms in the based category,

$$f, g : (A, G, a_0) \rightrightarrows (B, H, b_0).$$

Because $f(a_0) = b_0 = g(a_0)$, the basepoint is automatically an element on which $f$ and $g$ agree. Consequently the set-theoretic equalizer

$$E := \{a \in A : f(a) = g(a)\}$$

is nonempty, and indeed $a_0 \in E$. Hence $(E, G, a_0)$ is automatically an object of $\widehat{\mathrm{FR}}_k$, with the evident inclusion $E \hookrightarrow A$ a morphism preserving basepoints. In other words, the very condition that morphisms respect the basepoint forces the equalizer subset to contain the basepoint, so the obstruction encountered in $\mathrm{FR}_k$ (namely $E = \varnothing$) cannot occur. We regard this as the most basic instance of the "fallback element" role of the basepoint: whenever an otherwise-defined subobject might be empty, the basepoint condition forces at least one element to survive.

The same mechanism propagates to other limit constructions built from iterated equalizers. For instance, in forming a pullback of

$$(A, G, a_0) \xrightarrow{f} (C, K, c_0) \xleftarrow{g} (B, H, b_0),$$

we look at the set of pairs $(a, b) \in A \times B$ with $f(a) = g(b)$. Even before worrying about ambient groups, the basepoint condition gives a canonical pair $(a_0, b_0)$ satisfying $f(a_0) = c_0 = g(b_0)$. Thus the pullback subset is again automatically nonempty. More generally, any finite limit diagram in which each object carries a chosen point and each morphism preserves it has a distinguished compatible family of points; this family lies in the underlying set-theoretic limit and thereby witnesses nonemptiness. What fails in $\mathrm{FR}_k$ is not the abstract existence of set-theoretic limits, but the compatibility between those limits and the requirement that our objects be finite and nonempty; basepoints enforce that compatibility.

The basepoint is equally important for colimits, but for a different reason. Colimit constructions in $\mathrm{FR}_k$ suffer not only from potential empty identifications but also from non-functorial "placement" issues: without a distinguished element, there is no canonical way to position two additive sets inside a common ambient group without making arbitrary translation choices. In $\widehat{\mathrm{FR}}_k$ the basepoint provides a canonical anchor for gluing. Concretely, when we form a coproduct-like object, the role of the basepoint is to specify where the two components meet: we may insist that the images of the basepoints coincide, while the remaining elements are placed in a way that avoids introducing unintended $k$-term relations across components. This is formally analogous to wedge sums in pointed sets: we glue along the distinguished points and treat everything else as separate. The key categorical point is that the universal property then refers to maps that already preserve basepoints, so the gluing locus is forced and does not depend on arbitrary choices.

From this perspective, $\widehat{\mathrm{FR}}_k$ should be viewed as an "affine" analogue of the category of pointed sets. The basepoint is not an ambient-group zero; it is merely a marked element of the finite subset. Thus we do not rigidify an object by insisting it contain 0 (which would select a preferred translate), and we do not forbid translations (which is what happens if one demands that morphisms fix 0 as an ambient element). Instead, we record whatever element we wish to mark and require maps to respect that marking. This choice preserves the usual additive-combinatorial practice of treating $A$ and its translates as essentially equivalent, while still allowing us to perform categorical constructions functorially.

It is also useful to isolate the role of singleton objects. In $\widehat{\mathrm{FR}}_k$, any singleton $(\{a_0\}, G, a_0)$ admits exactly one morphism to any other based object $(B, H, b_0)$, namely the constant map sending $a_0$ to $b_0$. For $k \geq 2$ this map is a Freiman $k$-homomorphism because all $k$-term additive relations in a singleton are tautologically preserved, and basepoint preservation forces uniqueness. Dually, there is exactly one morphism from any $(A, G, a_0)$ to a singleton $(\{b_0\}, H, b_0)$. Thus singleton objects behave as zero objects in the pointed sense: they provide canonical absorbing targets and sources for maps, mirroring how the basepoint functions as a canonical "degenerate value" when configurations collapse.

We stress that while the basepoint prevents emptiness, it does not collapse the additive information of $A$. The underlying set $A$ still sits in its ambient group $G$, and Freiman $k$-homomorphisms still record precisely the $k$-term additive relations among elements of $A$. The only additional constraint is that each object carries a marked element and each morphism must send marked element to marked element. This mild rigidity is exactly what is needed to make the usual finite limit and colimit patterns nondegenerate and functorial: limits become nonempty because the compatible family of basepoints supplies an element of the limit, and colimits become canonical because the basepoint supplies a forced locus for identifications. Having set up $\widehat{\mathrm{FR}}_k$, we next describe a canonical way to pass from an unbased Freiman object $(A, G)$ to a based one, in a manner compatible with translations and suited to universal constructions.

# 3 3. The canonical completion functor $(-)^+ : \mathrm{FR}_k \to \widehat{\mathrm{FR}}_k$: explicit construction via $G \oplus \mathbb{Z}$, functoriality, faithfulness, and translation isomorphisms.

We now define a canonical "pointed completion" functor

$$(-)^+ : \mathrm{FR}_k \longrightarrow \widehat{\mathrm{FR}}_k$$

which adjoins to an unbased Freiman object a distinguished point in a way that is functorial and compatible with translations. The basic requirement is that the adjoined point be genuinely new (so that it can serve as a marked element for universal constructions) while remaining within an ambient abelian group.

Let $(A, G)$ be an object of $\mathrm{FR}_k$. We set

$$(A, G)^+ := \big(A \times \{0\} \ \cup \ \{\star\},\ G \oplus \mathbb{Z},\ \star\big), \qquad \star := (0, 1) \in G \oplus \mathbb{Z}.$$

We view $A$ as embedded into $G \oplus \mathbb{Z}$ via $a \mapsto (a, 0)$, and we adjoin the extra element $\star$ with nonzero $\mathbb{Z}$-coordinate. By construction $A \times \{0\} \cup \{\star\}$ is finite and nonempty, and $\star$ is a distinguished element. The choice $\star = (0, 1)$ is not essential up to canonical isomorphism, but it is convenient: the projection to the $\mathbb{Z}$-factor will control additive relations involving $\star$ in a way that is uniform across all objects.

The key point is that Freiman relations in $(A, G)^+$ split according to the number of occurrences of $\star$. Indeed, consider a $k$-term additive relation in the ambient group $G \oplus \mathbb{Z}$,

$$x_1 + \cdots + x_k = x'_1 + \cdots + x'_k, \qquad x_i, x'_i \in A \times \{0\} \cup \{\star\}.$$

Applying the projection $\pi_{\mathbb{Z}} : G \oplus \mathbb{Z} \to \mathbb{Z}$, and using that $\pi_{\mathbb{Z}}(a, 0) = 0$ and $\pi_{\mathbb{Z}}(\star) = 1$, we obtain

$$\#\{i : x_i = \star\} = \#\{i : x'_i = \star\}.$$

Thus a $k$-term relation can involve $\star$, but it must involve it the same number of times on both sides. After cancelling these $\star$-contributions (at the level of equality of sums in $G \oplus \mathbb{Z}$), the relation reduces to a $k'$-term relation among elements of $A \times \{0\}$, where $k' = k - m$ and $m$ is the common number of $\star$'s. In particular, relations not involving $\star$ are precisely the relations in the copy of $A \subseteq G$ sitting in degree 0.

Now let $f : (A, G) \to (B, H)$ be a morphism in $\mathrm{FR}_k$, i.e. a Freiman $k$-homomorphism $f : A \to B$. We define

$$f^+ : (A, G)^+ \longrightarrow (B, H)^+$$

by the formulas

$$f^+(a, 0) := (f(a), 0) \quad (a \in A), \qquad f^+(\star) := \star.$$

This is a well-defined set map $A \times \{0\} \cup \{\star\} \to B \times \{0\} \cup \{\star\}$, and it is basepoint-preserving by definition. It remains to check that it is a Freiman $k$-homomorphism (in the ambient groups $G \oplus \mathbb{Z}$ and $H \oplus \mathbb{Z}$).

**Lemma.** The map $f^+$ is a Freiman $k$-homomorphism.

*Proof.* Suppose

$$x_1 + \cdots + x_k = x_1' + \cdots + x_k' \qquad \text{in } G \oplus \mathbb{Z}, \quad x_i, x_i' \in A \times \{0\} \cup \{\star\}.$$

As above, comparing $\mathbb{Z}$-coordinates shows that $\star$ occurs the same number $m$ of times among the $x_i$ and the $x_i'$. Reordering terms if necessary, we may write

$$x_1 = \cdots = x_m = \star, \quad x_1' = \cdots = x_m' = \star,$$

and $x_{m+1}, \ldots, x_k \in A \times \{0\}$, $x_{m+1}', \ldots, x_k' \in A \times \{0\}$. Cancelling the common sum $\star + \cdots + \star$ (with $m$ terms) yields

$$x_{m+1} + \cdots + x_k = x_{m+1}' + \cdots + x_k' \qquad \text{in } G \oplus \mathbb{Z}.$$

Since the remaining terms have $\mathbb{Z}$-coordinate 0, this equality is equivalent to an equality in $G$ among the corresponding elements of $A$. Applying $f$ and using that $f$ is a Freiman $k$-homomorphism (hence also preserves all additive relations of length $\leq k$ obtained by repetition), we get the corresponding equality in $H$. Reintroducing the $m$ copies of $\star$ and observing that $f^+(\star) = \star$, we conclude

$$f^+(x_1) + \cdots + f^+(x_k) = f^+(x_1') + \cdots + f^+(x_k') \qquad \text{in } H \oplus \mathbb{Z},$$

as required. □

Thus $(-)^+$ is well-defined on morphisms. Functoriality is immediate from the defining formulas: for the identity $\mathrm{id}_A$ we have $(\mathrm{id}_A)^+ = \mathrm{id}_{(A,G)^+}$, and for a composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ we have $(g \circ f)^+ = g^+ \circ f^+$ because both sides agree on $A \times \{0\}$ and fix $\star$. Hence $(-)^+$ is a functor $\mathrm{FR}_k \to \widehat{\mathrm{FR}}_k$.

The functor $(-)^+$ is *faithful*. Indeed, if $f, g : (A, G) \rightrightarrows (B, H)$ satisfy $f^+ = g^+$, then for each $a \in A$ we have

$$(f(a), 0) = f^+(a, 0) = g^+(a, 0) = (g(a), 0),$$

so $f(a) = g(a)$ and hence $f = g$. In particular, $\mathrm{FR}_k$ identifies with a subcategory of $\widehat{\mathrm{FR}}_k$ up to the evident embedding on hom-sets. We emphasize that $(-)^+$ is not full in general: in $\widehat{\mathrm{FR}}_k$ there are additional basepoint-preserving Freiman maps $(A, G)^+ \to (B, H)^+$ that may collapse elements of $A \times \{0\}$ to $\star$, a phenomenon which cannot occur in $\mathrm{FR}_k$.

On objects, $(-)^+$ is injective on isomorphism classes in the following sense: if $\phi : (A, G)^+ \to (B, H)^+$ is an isomorphism in $\widehat{\mathrm{FR}}_k$, then $\phi(\star) = \star$ and, since $\phi$ is bijective, $\phi$ restricts to a bijection $A \times \{0\} \to B \times \{0\}$. Identifying $A$ with $A \times \{0\}$ and $B$ with $B \times \{0\}$, this restriction is a Freiman $k$-isomorphism $A \to B$. Thus isomorphisms between pointed completions do not identify genuinely different unbased objects; they only recover the expected notion of Freiman isomorphism on the underlying sets.

Finally, the completion functor is designed so that translations become canonical based isomorphisms. Let $A \subseteq G$ and $g \in G$. The translation map $\tau_g : A \to A + g$, $a \mapsto a + g$, is a Freiman $k$-isomorphism in $\mathrm{FR}_k$. Applying $(-)^+$ yields a basepoint-preserving map

$$\tau_g^+ : (A, G)^+ \longrightarrow (A + g, G)^+, \qquad \tau_g^+(a, 0) = (a + g, 0), \quad \tau_g^+(\star) = \star,$$

which is an isomorphism with inverse $\tau_{-g}^+$. Since both completions use the same ambient group $G \oplus \mathbb{Z}$ and the same basepoint $\star = (0, 1)$, this identification is canonical and does not depend on any choice of where the translate should "sit". In particular, different translates of $A$ become canonically isomorphic in $\widehat{\mathrm{FR}}_k$ via basepoint-preserving morphisms, a feature that will be used repeatedly when comparing colimit constructions across different ambient placements.

### 3.1 Finite limits in $\widehat{\mathrm{FR}}_k$

We next verify that $\widehat{\mathrm{FR}}_k$ is finitely complete by exhibiting explicit models for the terminal object, binary products, equalizers, and pullbacks. Since finite limits can be built from products and equalizers, it suffices to treat these constructions and record their interaction with the basepoint condition.

**Terminal object.** Let 0 denote the trivial abelian group. We claim that

$$\mathbf{1} := (\{0\}, 0, 0)$$

is terminal in $\widehat{\mathrm{FR}}_k$. Indeed, given any $(A, G, a_0)$ there is exactly one basepoint-preserving set map $A \to \{0\}$, namely the constant map $a \mapsto 0$, and it is a

Freiman $k$-homomorphism because any $k$-term additive relation in $A$ is sent to the tautological equality $0 + \cdots + 0 = 0 + \cdots + 0$ in $0$. Uniqueness is immediate. (Any pointed singleton $(\{b_0\}, H, b_0)$ is likewise terminal, hence canonically isomorphic to $\mathbf{1}$.)

**Binary products.** Given objects $(A, G, a_0)$ and $(B, H, b_0)$, we define their product to be

$$(A, G, a_0) \times (B, H, b_0) := \big(A \times B,\ G \oplus H,\ (a_0, b_0)\big),$$

where $A \times B$ is viewed as a subset of $G \oplus H$ via $(a, b) \mapsto (a, b)$. The projections are the evident maps

$$\pi_A : A \times B \to A, \quad \pi_A(a, b) = a, \qquad \pi_B : A \times B \to B, \quad \pi_B(a, b) = b,$$

which preserve basepoints.

We check that $\pi_A$ is a Freiman $k$-homomorphism (and similarly $\pi_B$). Suppose

$$(a_1, b_1) + \cdots + (a_k, b_k) = (a_1', b_1') + \cdots + (a_k', b_k') \qquad \text{in } G \oplus H$$

with all $(a_i, b_i), (a_i', b_i') \in A \times B$. Equality in the direct sum means equality in each coordinate, hence

$$a_1 + \cdots + a_k = a_1' + \cdots + a_k' \quad \text{in } G \qquad \text{and} \qquad b_1 + \cdots + b_k = b_1' + \cdots + b_k' \quad \text{in } H.$$

Applying $\pi_A$ simply extracts the first coordinate, so the required relation in $A$ holds. Thus $\pi_A$ and $\pi_B$ are morphisms in $\widehat{\mathrm{FR}}_k$.

Now let $(C, K, c_0)$ be any object and let

$$f : (C, K, c_0) \to (A, G, a_0), \qquad g : (C, K, c_0) \to (B, H, b_0)$$

be morphisms. Define $\langle f, g \rangle : C \to A \times B$ by

$$\langle f, g \rangle(c) := (f(c), g(c)).$$

This map preserves basepoints and satisfies $\pi_A \circ \langle f, g \rangle = f$ and $\pi_B \circ \langle f, g \rangle = g$. To see that $\langle f, g \rangle$ is Freiman, take a $k$-term relation $c_1 + \cdots + c_k = c_1' + \cdots + c_k'$ in $C$ (in the Freiman sense, i.e. among elements of $C \subseteq K$). Since $f$ and $g$ are Freiman $k$-homomorphisms, we have

$$f(c_1) + \cdots + f(c_k) = f(c_1') + \cdots + f(c_k') \quad \text{in } G, \qquad g(c_1) + \cdots + g(c_k) = g(c_1') + \cdots + g(c_k') \quad \text{in } H.$$

Combining these equalities gives

$$\langle f, g \rangle(c_1) + \cdots + \langle f, g \rangle(c_k) = \langle f, g \rangle(c_1') + \cdots + \langle f, g \rangle(c_k') \quad \text{in } G \oplus H,$$

so $\langle f, g \rangle$ is a morphism. Uniqueness follows from the set-theoretic uniqueness of a map into a Cartesian product determined by its compositions with the projections. Hence the above object is the categorical product in $\widehat{\mathrm{FR}}_k$.

**Equalizers.** Let $f, g : (A, G, a_0) \rightrightarrows (B, H, b_0)$ be parallel morphisms in $\widehat{\mathrm{FR}}_k$. We define the equalizer object to be

$$\mathrm{Eq}(f, g) := (E, G, a_0), \qquad E := \{a \in A : \ f(a) = g(a)\} \subseteq G,$$

with structure map the inclusion $i : E \hookrightarrow A$ (basepoint-preserving). This is well-defined because $f(a_0) = b_0 = g(a_0)$, so $a_0 \in E$, and thus $E$ is nonempty.

The inclusion $i$ is a Freiman $k$-homomorphism: if $e_1 + \cdots + e_k = e_1' + \cdots + e_k'$ holds in $G$ with $e_i, e_i' \in E \subseteq A$, then the same equality holds in $A \subseteq G$, and applying $i$ does nothing. Moreover, $f \circ i = g \circ i$ by definition of $E$.

For the universal property, suppose $h : (C, K, c_0) \to (A, G, a_0)$ satisfies $f \circ h = g \circ h$. Then for each $c \in C$ we have $f(h(c)) = g(h(c))$, hence $h(c) \in E$, so $h$ factors uniquely as a set map through the inclusion $i$, say $h = i \circ \widetilde{h}$ with $\widetilde{h} : C \to E$. Since $i$ is injective and $\widetilde{h}$ is just $h$ with codomain restricted, $\widetilde{h}$ inherits the Freiman property from $h$: any $k$-term relation in $C$ mapped by $h$ lands in $E$ and remains valid in $G$. Thus $\widetilde{h}$ is a morphism in $\widehat{\mathrm{FR}}_k$, and uniqueness is clear. Hence $(E, G, a_0)$ is an equalizer.

**Pullbacks.** Consider a cospan in $\widehat{\mathrm{FR}}_k$,

$$(A, G, a_0) \xrightarrow{f} (C, K, c_0) \xleftarrow{g} (B, H, b_0).$$

We define its pullback to be

$$(A, G, a_0) \times_{(C,K,c_0)} (B, H, b_0) := (P,\ G \oplus H,\ (a_0, b_0)),$$

where

$$P := \{(a, b) \in A \times B : \ f(a) = g(b)\} \subseteq G \oplus H.$$

This subset is finite, and it is nonempty because $f(a_0) = c_0 = g(b_0)$ implies $(a_0, b_0) \in P$.

Let $p_A : P \to A$ and $p_B : P \to B$ be the restrictions of the product projections. The same coordinate argument as for products shows that $p_A$ and $p_B$ are Freiman $k$-homomorphisms. By construction we have $f \circ p_A = g \circ p_B$ as set maps (hence as morphisms).

Now let $(T, L, t_0)$ be an object and suppose we have morphisms

$$u : (T, L, t_0) \to (A, G, a_0), \qquad v : (T, L, t_0) \to (B, H, b_0)$$

with $f \circ u = g \circ v$. Define $w : T \to P$ by $w(t) := (u(t), v(t))$. The compatibility $f \circ u = g \circ v$ ensures $w(t) \in P$ for all $t$, and $w$ preserves basepoints. To check the Freiman property, take a $k$-term relation $t_1 + \cdots + t_k = t_1' + \cdots + t_k'$ in $T$. Applying $u$ and $v$ yields the corresponding relations in $G$ and $H$, and hence the relation

$$w(t_1) + \cdots + w(t_k) = w(t_1') + \cdots + w(t_k') \quad \text{in } G \oplus H,$$

so $w$ is a morphism. Uniqueness follows because any map $T \to P \subseteq A \times B$ is determined by its compositions with $p_A$ and $p_B$. Thus the above construction is a pullback.

**Consequences and comparison with $\mathrm{FR}_k$.** We have therefore exhibited a terminal object, binary products, and equalizers in $\widehat{\mathrm{FR}}_k$, and hence $\widehat{\mathrm{FR}}_k$ has all finite limits. The role of basepoints is not cosmetic: it guarantees nonemptiness of equalizers and pullbacks. In the unbased category $\mathrm{FR}_k$, the set-theoretic equalizer of two Freiman maps $A \rightrightarrows B$ can be empty, and since objects are required to be nonempty, the equalizer need not exist as an object of $\mathrm{FR}_k$. Likewise, given a cospan $A \to C \leftarrow B$ in $\mathrm{FR}_k$, the fiber product $\{(a, b) : f(a) = g(b)\}$ can be empty, obstructing pullbacks. In $\widehat{\mathrm{FR}}_k$, the condition that all morphisms preserve basepoints forces $a_0$ and $b_0$ to map to the same $c_0$, producing a canonical point $(a_0, b_0)$ in the pullback and a canonical point $a_0$ in the equalizer whenever the diagram is parallel. This is the basic mechanism by which finite completeness is restored without imposing a normalization such as $0 \in A$ and $f(0) = 0$, and it will be crucial when we pass to finite colimits.

### 3.2 Finite colimits in $\widehat{\mathrm{FR}}_k$

We next verify that $\widehat{\mathrm{FR}}_k$ is finitely cocomplete by exhibiting explicit models for the initial object, binary coproducts, coequalizers, and pushouts. Since finite colimits can be built from coproducts and coequalizers, it suffices to treat these constructions and record how the basepoint condition ensures that the resulting objects remain nonempty and that the universal properties are witnessed by basepoint-preserving Freiman maps.

**Initial object.** The terminal object $\mathbf{1} = (\{0\}, 0, 0)$ constructed above is also initial. Indeed, for any $(A, G, a_0)$ there is exactly one basepoint-preserving map $\{0\} \to A$, namely $0 \mapsto a_0$, and it is a Freiman $k$-homomorphism for the same tautological reason as before. Thus $\widehat{\mathrm{FR}}_k$ is pointed (it has a zero object).

**Binary coproducts as wedges.** Let $(A, G, a_0)$ and $(B, H, b_0)$ be objects. We define their coproduct to be the wedge obtained by identifying basepoints and keeping the two summands otherwise disjoint at the level of the underlying finite subset. Set

$$\widetilde{L} := G \oplus H \oplus \mathbb{Z}, \qquad r := (a_0, -b_0, -1) \in \widetilde{L}, \qquad L := \widetilde{L}/\langle r \rangle,$$

and let $\overline{(\cdot)} : \widetilde{L} \to L$ be the quotient map. Define a finite subset

$$W := \{\overline{(a, 0, 0)} : \; a \in A\} \; \cup \; \{\overline{(0, b, 1)} : \; b \in B\} \; \subseteq L$$

and take the basepoint to be

$$w_0 := \overline{(a_0,0,0)} = \overline{(0,b_0,1)} \in W.$$

We write

$$(A,G,a_0) \vee (B,H,b_0) := (W,L,w_0).$$

There are canonical maps

$$i_A : A \to W, \quad i_A(a) = \overline{(a,0,0)}, \qquad i_B : B \to W, \quad i_B(b) = \overline{(0,b,1)},$$

which preserve basepoints by construction. The map $i_A$ is a Freiman $k$-homomorphism because it is the restriction of a group homomorphism $\widetilde{L} \to L$ to the subset $A \times \{0\} \times \{0\}$ (and similarly for $i_B$). Moreover, the only identification between the two displayed subsets of $L$ is at the basepoint: if $\overline{(a,0,0)} = \overline{(0,b,1)}$, then $(a,-b,-1) \in \langle r \rangle$, forcing $(a,-b,-1) = r$ and hence $a = a_0$ and $b = b_0$.

To verify the universal property, let $(C,K,c_0)$ be an object and let

$$f : (A,G,a_0) \to (C,K,c_0), \qquad g : (B,H,b_0) \to (C,K,c_0)$$

be morphisms. Since $f(a_0) = c_0 = g(b_0)$ and since $i_A(A) \cap i_B(B) = \{w_0\}$, there is a unique basepoint-preserving set map $h : W \to C$ such that $h \circ i_A = f$ and $h \circ i_B = g$, namely

$$h\big(\overline{(a,0,0)}\big) := f(a), \qquad h\big(\overline{(0,b,1)}\big) := g(b).$$

It remains to show that $h$ is Freiman of order $k$. Let

$$x_1 + \cdots + x_k = x'_1 + \cdots + x'_k \qquad \text{in } L$$

with all $x_i, x'_i \in W$. Choose lifts $\widetilde{x}_i, \widetilde{x}'_i \in \widetilde{L}$ of the form $(a,0,0)$ or $(0,b,1)$. Then

$$\sum_{i=1}^{k} \widetilde{x}_i - \sum_{i=1}^{k} \widetilde{x}'_i = n\,r$$

for some $n \in \mathbb{Z}$. Let $m$ (resp. $m'$) be the number of $B$-type terms among the $\widetilde{x}_i$ (resp. among the $\widetilde{x}'_i$). Comparing the $\mathbb{Z}$-coordinates gives $m - m' = -n$, i.e. $n = m' - m$. Writing the $G$-coordinate equality and rearranging using $n = m' - m$ yields a $k$-term relation in $A$:

$$\sum_{x_i = i_A(a_i)} a_i \;+\; m\,a_0 \;=\; \sum_{x'_i = i_A(a'_i)} a'_i \;+\; m'\,a_0 \qquad \text{in } G,$$

where the displayed sums over $a_i$ and $a'_i$ involve exactly $k - m$ and $k - m'$ terms respectively, and the remaining $m$ and $m'$ terms are filled by the basepoint $a_0$. Since $f$ is a Freiman $k$-homomorphism, applying $f$ gives

$$\sum_{x_i = i_A(a_i)} f(a_i) \;+\; m\,c_0 \;=\; \sum_{x'_i = i_A(a'_i)} f(a'_i) \;+\; m'\,c_0 \qquad \text{in } K.$$

Similarly, from the $H$-coordinate equality we obtain a $k$-term relation in $B$ which, after applying $g$, yields

$$\sum_{x_i=i_B(b_i)} g(b_i) \;+\; m' \, c_0 \;=\; \sum_{x'_i=i_B(b'_i)} g(b'_i) \;+\; m \, c_0 \qquad \text{in } K.$$

Adding these two equalities cancels the basepoint contributions and gives

$$h(x_1) + \cdots + h(x_k) = h(x'_1) + \cdots + h(x'_k) \qquad \text{in } K,$$

which is exactly the Freiman condition for $h$. Uniqueness of $h$ as a morphism follows from uniqueness as a set map. Hence $(W, L, w_0)$ is the coproduct of $(A, G, a_0)$ and $(B, H, b_0)$ in $\widehat{\mathrm{FR}}_k$. (Finite coproducts follow by iteration.)

**Coequalizers.** Let $f, g : (A, G, a_0) \rightrightarrows (B, H, b_0)$ be parallel morphisms. We construct a coequalizer by presenting an ambient abelian group in which we impose, as actual equalities, both the $k$-term additive relations already holding in $B \subseteq H$ and the additional identifications $f(a) = g(a)$ for $a \in A$. Let

$$F_B := \bigoplus_{b \in B} \mathbb{Z} \, e_b$$

be the free abelian group on the underlying set $B$. Let $N \leq F_B$ be the subgroup generated by

$$e_{b_1} + \cdots + e_{b_k} - e_{b'_1} - \cdots - e_{b'_k} \quad \text{whenever} \quad b_1 + \cdots + b_k = b'_1 + \cdots + b'_k \text{ in } H,$$

together with the generators $e_{f(a)} - e_{g(a)}$ for all $a \in A$. Put $\overline{H} := F_B/N$ and let $\overline{(\,\cdot\,)} : F_B \to \overline{H}$ be the quotient map. Define a finite subset

$$\overline{B} := \{\overline{e_b} : \ b \in B\} \subseteq \overline{H}, \qquad \bar{b}_0 := \overline{e_{b_0}} \in \overline{B},$$

and let

$$\mathrm{Coeq}(f, g) := (\overline{B}, \overline{H}, \bar{b}_0), \qquad q : (B, H, b_0) \to (\overline{B}, \overline{H}, \bar{b}_0), \quad q(b) = \overline{e_b}.$$

By construction, $q$ is basepoint-preserving and coequalizes $f$ and $g$. It is a Freiman $k$-homomorphism because every $k$-term relation holding in $B$ becomes an equality among the corresponding elements $\overline{e_b}$ in $\overline{H}$. For the universal property, let $h : (B, H, b_0) \to (C, K, c_0)$ be a morphism with $h \circ f = h \circ g$. There is a unique group homomorphism $\phi : F_B \to K$ with $\phi(e_b) = h(b)$. The Freiman property of $h$ implies that $\phi$ kills each generator coming from a $k$-term relation in $B$, and the condition $h \circ f = h \circ g$ implies that $\phi(e_{f(a)} - e_{g(a)}) = 0$ for all $a$. Hence $N \subseteq \ker(\phi)$, so $\phi$ factors uniquely through a homomorphism $\overline{\phi} : \overline{H} \to K$. Restricting $\overline{\phi}$ to $\overline{B}$ gives a unique morphism $\widetilde{h} : (\overline{B}, \overline{H}, \bar{b}_0) \to (C, K, c_0)$ with $\widetilde{h} \circ q = h$. Thus $q$ is a coequalizer in $\widehat{\mathrm{FR}}_k$.

**Pushouts.** Given a span

$$(A, G, a_0) \xleftarrow{f} (C, K, c_0) \xrightarrow{g} (B, H, b_0)$$

we define its pushout by the standard recipe: form the wedge $(A, G, a_0) \vee (B, H, b_0)$ with structure maps $i_A, i_B$, and then take the coequalizer of the parallel pair

$$i_A \circ f,\ i_B \circ g : (C, K, c_0) \rightrightarrows (A, G, a_0) \vee (B, H, b_0).$$

Since we have constructed coproducts and coequalizers explicitly, this yields a concrete pushout object in $\widehat{\mathrm{FR}}_k$, together with the induced maps from $(A, G, a_0)$ and $(B, H, b_0)$, and the usual universal property follows formally from the universal properties already verified.

**Subgroup-closure and comparison with $\mathrm{FR}_k$.** A salient point in the above constructions is that the ambient group of a colimit is obtained as an explicit quotient of a direct sum or a free abelian group by a subgroup generated by relations; in particular, no additional "subgroup closure" step is required to make the universal identifications compatible with the abelian-group structure. This is in contrast with naive set-level gluings inside a fixed ambient group, where one typically has to enlarge the ambient group to accommodate the imposed identifications as actual additive equalities. The based condition is again essential at the level of existence statements: $\mathrm{FR}_k$ has no initial object (since objects are required to be nonempty), and without a distinguished point there is no canonical wedge-type coproduct compatible with the Freiman structure. In $\widehat{\mathrm{FR}}_k$, the basepoint both supplies the necessary nonemptiness and provides a canonical locus along which coproducts and pushouts are formed, restoring finite cocompleteness in a manner compatible with translation symmetries.

**Reflective embedding of $\mathrm{FR}_k^0$.** We record the precise relationship between the normalized Freiman category $\mathrm{FR}_k^0$ and the based category $\widehat{\mathrm{FR}}_k$. Recall that an object of $\mathrm{FR}_k^0$ is a pair $(A, G)$ with $0 \in A \subseteq G$, and morphisms are Freiman $k$-homomorphisms preserving 0. There is an evident inclusion functor

$$J : \mathrm{FR}_k^0 \hookrightarrow \widehat{\mathrm{FR}}_k, \qquad J(A, G) := (A, G, 0),$$

acting as the identity on the underlying set maps. In particular, if $f : (A, G) \to (B, H)$ is a morphism in $\mathrm{FR}_k^0$, then $f(0) = 0$, and the same set map defines a morphism $J(f) : (A, G, 0) \to (B, H, 0)$ in $\widehat{\mathrm{FR}}_k$.

We first note that $J$ is fully faithful. Faithfulness is immediate since $J$ does not change the underlying set map. For fullness, let $\varphi : (A, G, 0) \to (B, H, 0)$ be a morphism in $\widehat{\mathrm{FR}}_k$. Then $\varphi$ is a Freiman $k$-homomorphism $A \to B$ satisfying $\varphi(0) = 0$, hence it is by definition a morphism $(A, G) \to$

$(B, H)$ in $\mathrm{FR}_k^0$. Thus every morphism in $\widehat{\mathrm{FR}}_k$ between objects in the image of $J$ comes from a unique morphism in $\mathrm{FR}_k^0$, proving that $J$ is fully faithful.

We next define a normalization (or re-basing) functor

$$R : \widehat{\mathrm{FR}}_k \to \mathrm{FR}_k^0, \qquad R(A, G, a_0) := (A - a_0,\ G),$$

where $A - a_0 := \{a - a_0 :\ a \in A\} \subseteq G$. This is well-defined since $A$ is finite and nonempty and $0 = a_0 - a_0 \in A - a_0$, so $R(A, G, a_0)$ is indeed an object of $\mathrm{FR}_k^0$. On morphisms, if

$$f : (A, G, a_0) \to (B, H, b_0)$$

is a morphism in $\widehat{\mathrm{FR}}_k$, we define $R(f) : A - a_0 \to B - b_0$ by the formula

$$R(f)(a - a_0) := f(a) - b_0 \qquad (a \in A).$$

This is well-defined as a set map since for each $a \in A$ we have $f(a) \in B$, hence $f(a) - b_0 \in B - b_0$. It also preserves 0, because $R(f)(0) = R(f)(a_0 - a_0) = f(a_0) - b_0 = b_0 - b_0 = 0$. It remains to check that $R(f)$ is a Freiman $k$-homomorphism. Suppose

$$(x_1) + \cdots + (x_k) = (x_1') + \cdots + (x_k') \qquad \text{in } G,$$

with each $x_i = a_i - a_0$ and each $x_i' = a_i' - a_0$ for some $a_i, a_i' \in A$. Adding $ka_0$ to both sides yields

$$a_1 + \cdots + a_k = a_1' + \cdots + a_k' \qquad \text{in } G.$$

Since $f$ is Freiman of order $k$, we obtain

$$f(a_1) + \cdots + f(a_k) = f(a_1') + \cdots + f(a_k') \qquad \text{in } H.$$

Subtracting $kb_0$ from both sides gives

$$\big(f(a_1) - b_0\big) + \cdots + \big(f(a_k) - b_0\big) = \big(f(a_1') - b_0\big) + \cdots + \big(f(a_k') - b_0\big),$$

which is exactly the Freiman condition for $R(f)$ on $A - a_0$. Thus $R$ is a well-defined functor $\widehat{\mathrm{FR}}_k \to \mathrm{FR}_k^0$.

We now verify that $R$ is left adjoint to $J$. Concretely, for each $(A, G, a_0) \in \widehat{\mathrm{FR}}_k$ and each $(B, H) \in \mathrm{FR}_k^0$ we claim there is a natural bijection

$$\mathrm{Hom}_{\mathrm{FR}_k^0}\big(R(A, G, a_0), (B, H)\big) \ \cong\ \mathrm{Hom}_{\widehat{\mathrm{FR}}_k}\big((A, G, a_0), J(B, H)\big).$$

Given a morphism $\psi : (A - a_0, G) \to (B, H)$ in $\mathrm{FR}_k^0$, we define a morphism $\widetilde{\psi} : (A, G, a_0) \to (B, H, 0)$ in $\widehat{\mathrm{FR}}_k$ by

$$\widetilde{\psi}(a) := \psi(a - a_0) \qquad (a \in A).$$

This preserves basepoints since $\widetilde{\psi}(a_0) = \psi(0) = 0$, and it is Freiman of order $k$ because $a_1+\cdots+a_k = a'_1+\cdots+a'_k$ implies $(a_1-a_0)+\cdots+(a_k-a_0) = (a'_1-a_0)+\cdots+(a'_k-a_0)$, so the Freiman property of $\psi$ yields the required equality of sums in $H$. Conversely, given a morphism $\varphi : (A, G, a_0) \to (B, H, 0)$ in $\widehat{\mathrm{FR}}_k$, we define $\widehat{\varphi} : A - a_0 \to B$ by

$$\widehat{\varphi}(a - a_0) := \varphi(a) \qquad (a \in A).$$

This is well-defined as a set map $A - a_0 \to B$, it preserves 0 since $\widehat{\varphi}(0) = \varphi(a_0) = 0$, and it is Freiman of order $k$ by the same translation argument as above. These two assignments are inverse to each other and are natural in both variables, giving the desired adjunction $R \dashv J$.

For later use it is convenient to write the unit and counit explicitly. For an object $(A, G, a_0)$ of $\widehat{\mathrm{FR}}_k$, the unit

$$\eta_{(A,G,a_0)} : (A, G, a_0) \longrightarrow JR(A, G, a_0) = (A - a_0, G, 0)$$

is the translation map $\eta_{(A,G,a_0)}(a) = a - a_0$. It is basepoint-preserving and is a Freiman $k$-homomorphism as it is induced by the group homomorphism $G \to G$, $x \mapsto x - a_0$ restricted to $A$. For an object $(B, H)$ of $\mathrm{FR}^0_k$, we have

$$RJ(B, H) = R(B, H, 0) = (B - 0, H) = (B, H),$$

so the counit $\varepsilon_{(B,H)} : RJ(B, H) \to (B, H)$ is the identity. The triangle identities reduce to the tautological equalities expressing that translating by $-a_0$ and then re-including does nothing further, and that normalizing an already normalized object is the identity.

This adjunction isolates the role of the basepoint: $\widehat{\mathrm{FR}}_k$ remembers a chosen point $a_0 \in A$, while $\mathrm{FR}^0_k$ forces that point to be 0 and admits only maps compatible with that normalization. The reflector $R$ implements the canonical normalization by translating $A$ so that $a_0$ becomes 0. In particular, any map in $\widehat{\mathrm{FR}}_k$ can be transported to a map in the normalized category by subtracting the image of the basepoint, which is precisely the formula defining $R(f)$. Thus $\mathrm{FR}^0_k$ identifies as a full reflective subcategory of $\widehat{\mathrm{FR}}_k$, and the distinction between the two is exactly whether one chooses to rigidify the translation degree of freedom by fixing the basepoint at 0.

**Universal property of the pointed completion.** We now justify the universal property stated in (v), which may be read as saying that $\widehat{\mathrm{FR}}_k$ is the "finite limit–finite colimit completion" of $\mathrm{FR}_k$ after forcing all singleton objects to behave as a single initial object. The role of the hypothesis on singletons is not cosmetic: in $\mathrm{FR}_k$ the objects $(\{x\}, G)$ are terminal (there is a unique map from any $(A, G)$ to $(\{x\}, G)$), but they are far from initial since there are many distinct maps $(\{x\}, G) \to (B, H)$, one for each choice of element of $B$. In contrast, in $\widehat{\mathrm{FR}}_k$ any singleton based object $(\{b_0\}, H, b_0)$

is a *zero object*: there is a unique basepoint-preserving map $(\{b_0\}, H, b_0) \to (B, H, b_0)$ and also a unique map $(B, H, b_0) \to (\{b_0\}, H, b_0)$, the latter being constant. Thus, when we pass from $\mathrm{FR}_k$ to $\widehat{\mathrm{FR}}_k$, all of the "many maps out of a singleton" are collapsed to the unique map out of an initial object. The condition in (v) exactly demands that $F$ already performs this collapse in $\mathcal{D}$, so that there is no obstruction to extending $F$ across the completion.

To make this precise, fix a finitely complete and finitely cocomplete category $\mathcal{D}$, and a functor $F : \mathrm{FR}_k \to \mathcal{D}$ such that:

1. for every singleton object $(\{x\}, G)$ in $\mathrm{FR}_k$, the object $F(\{x\}, G)$ is initial in $\mathcal{D}$, and
2. for every morphism $\alpha : (\{x\}, G) \to (\{y\}, H)$, we have $F(\alpha) = \mathrm{id}$.

By initiality, any two initial objects in $\mathcal{D}$ are canonically isomorphic, and by (2) these canonical identifications are coherent on the nose on the full subcategory of $\mathrm{FR}_k$ spanned by singletons. We therefore fix, once and for all, an initial object $0_{\mathcal{D}}$ of $\mathcal{D}$ and treat each $F(\{x\}, G)$ as *specified* to be (canonically) $0_{\mathcal{D}}$.

The key structural input is that $\widehat{\mathrm{FR}}_k$ is generated, under finite limits and colimits, by the essential image of the pointed completion functor $(-)^+ : \mathrm{FR}_k \to \widehat{\mathrm{FR}}_k$. Concretely, two elementary observations suffice. First, singleton based objects already arise from $(-)^+$ by a finite colimit. Indeed, for a singleton $(\{x\}, G)$, the object $(\{x\}, G)^+$ has underlying set $\{(x, 0), \star\}$ with basepoint $\star$. Consider the endomorphisms

$$\mathrm{id},\, c : (\{x\}, G)^+ \longrightarrow (\{x\}, G)^+,$$

where $c(\star) = \star$ and $c(x, 0) = \star$. (This is a morphism in $\widehat{\mathrm{FR}}_k$ since it is basepoint-preserving, and for $k \geq 2$ it is Freiman: all $k$-term sums in the source have the same image because $c$ is constant on the non-basepoint element.) The coequalizer of id and $c$ identifies $(x, 0)$ with $\star$ and is therefore canonically isomorphic to the singleton based object $(\{x\}, G, x)$. In symbols,

$$(\{x\}, G, x) \;\cong\; \mathrm{Coeq}\big(\mathrm{id}, c : (\{x\}, G)^+ \rightrightarrows (\{x\}, G)^+\big).$$

Second, an arbitrary based object is obtained from a $+$-object by forcing the adjoined basepoint $\star$ to coincide with the chosen basepoint $a_0 \in A$, again by a finite colimit. Let $(A, G, a_0)$ be an object of $\widehat{\mathrm{FR}}_k$, and let $i : (\{a_0\}, G) \to (A, G)$ denote the evident inclusion morphism in $\mathrm{FR}_k$. Passing to $(-)^+$ yields a morphism

$$i^+ : (\{a_0\}, G)^+ \longrightarrow (A, G)^+.$$

Let $q : (\{a_0\}, G)^+ \to (\{a_0\}, G, a_0)$ be the canonical morphism in $\widehat{\mathrm{FR}}_k$ that collapses both $(a_0, 0)$ and $\star$ to $a_0$ (equivalently, the coequalizer map described above). Then $(A, G, a_0)$ is (canonically) the pushout of the span $i^+$ and $q$:

$$(A, G, a_0) \;\cong\; \mathrm{Pushout}\Big((A, G)^+ \xleftarrow{\;i^+\;} (\{a_0\}, G)^+ \xrightarrow{\;q\;} (\{a_0\}, G, a_0)\Big).$$

Intuitively, we start with $(A, G)^+$, where $\star$ is the basepoint, and we glue in the singleton based object so as to identify $\star$ with the element $(a_0, 0) \in A \times \{0\}$; the result is precisely $A$ with basepoint $a_0$.

These two presentations show that every object of $\widehat{\mathrm{FR}}_k$ can be built from objects in the image of $(-)^+$ using finite colimits, and hence using both finite limits and colimits (since we already know $\widehat{\mathrm{FR}}_k$ is finitely bicomplete). Moreover, the additional morphisms in $\widehat{\mathrm{FR}}_k$ (those not coming from $\mathrm{FR}_k$ via $(-)^+$) are exactly the universal arrows attached to these finite (co)limit constructions; thus, any functor out of $\widehat{\mathrm{FR}}_k$ that preserves finite (co)limits is forced to take these morphisms to the corresponding universal arrows in the target.

We now construct the extension $\widehat{F} : \widehat{\mathrm{FR}}_k \to \mathcal{D}$. On the full subcategory $\mathrm{Im}((-)^+) \subseteq \widehat{\mathrm{FR}}_k$ we set

$$\widehat{F}\big((A, G)^+\big) := F(A, G), \qquad \widehat{F}(f^+) := F(f).$$

Next, we define $\widehat{F}$ on a singleton based object $(\{x\}, G, x)$ by using the coequalizer presentation above:

$$\widehat{F}(\{x\}, G, x) := \mathrm{Coeq}\big(\widehat{F}(\mathrm{id}), \widehat{F}(c) : F(\{x\}, G) \rightrightarrows F(\{x\}, G)\big).$$

Here $F(\{x\}, G) \cong 0_{\mathcal{D}}$ is initial, so there is a unique endomorphism; hence $\widehat{F}(\mathrm{id}) = \widehat{F}(c) = \mathrm{id}_{0_{\mathcal{D}}}$, and the coequalizer is canonically $0_{\mathcal{D}}$. This is exactly where the singleton hypothesis is used: it guarantees that *every* morphism of the form "collapse to a singleton" becomes the unique arrow dictated by initiality, so that the finite colimit prescriptions are compatible with the relations in $\widehat{\mathrm{FR}}_k$.

Finally, for a general based object $(A, G, a_0)$ we define $\widehat{F}(A, G, a_0)$ as the pushout in $\mathcal{D}$ of the image of the pushout span above:

$$\widehat{F}(A, G, a_0) := \mathrm{Pushout}\Big(F(A, G) \xleftarrow{\;F(i)\;} F(\{a_0\}, G) \to \widehat{F}(\{a_0\}, G, a_0)\Big).$$

Since both $F(\{a_0\}, G)$ and $\widehat{F}(\{a_0\}, G, a_0)$ are canonically $0_{\mathcal{D}}$, this pushout is canonically isomorphic to $F(A, G)$; however, we emphasize that we *define* $\widehat{F}(A, G, a_0)$ by the universal property of the pushout. Doing so ensures that $\widehat{F}$ automatically carries the specified pushout squares in $\widehat{\mathrm{FR}}_k$ to pushout squares in $\mathcal{D}$, and similarly for the coequalizer presentations of singleton based objects.

At this point, $\widehat{F}$ is defined on a generating class of objects and on the structural morphisms arising from the above coequalizers and pushouts. We then extend to all of $\widehat{\mathrm{FR}}_k$ by requiring that $\widehat{F}$ preserve the explicitly constructed finite limits and colimits of (i): for example, we set $\widehat{F}(X \times Y)$ to be $\widehat{F}(X) \times \widehat{F}(Y)$ using the chosen binary product in $\mathcal{D}$, define $\widehat{F}(\mathrm{Eq}(f,g))$ to be $\mathrm{Eq}(\widehat{F}(f), \widehat{F}(g))$, and similarly for coproducts, coequalizers, pullbacks, and pushouts. The only nontrivial verification is that this prescription is well-defined on morphisms and compatible with composition; but this follows from the uniqueness clauses in the universal properties of finite (co)limits, together with the fact that the constructions in (i) were functorial in the usual sense (any map into a limit or out of a colimit is determined by its composites with the legs of the cone or cocone).

Uniqueness is of the same nature. Suppose $\widehat{F}_1, \widehat{F}_2 : \widehat{\mathrm{FR}}_k \to \mathcal{D}$ preserve finite limits and colimits and are equipped with natural isomorphisms $\widehat{F}_j \circ (-)^+ \cong F$. Since every object of $\widehat{\mathrm{FR}}_k$ can be assembled from $+$-objects and singleton based objects by finitely many (co)limit operations, preservation of those operations forces $\widehat{F}_1$ and $\widehat{F}_2$ to agree on objects and morphisms up to a unique isomorphism, obtained recursively by transporting along the universal properties. The resulting comparison isomorphisms are unique because any two maps between (co)limits agreeing on the legs of the defining cones/cocones are equal. Thus the extension exists and is unique up to unique isomorphism.

Finally, we stress the "up to isomorphism" aspect: unless one works in a setting where finite limits and colimits are *chosen* (so that preservation can be demanded strictly), the best one can ask for in ordinary category theory is uniqueness up to unique natural isomorphism. This is exactly the level of strictness used throughout: the constructions in (i) provide concrete models of limits and colimits in $\widehat{\mathrm{FR}}_k$, but any other choice is canonically isomorphic, and the extension $\widehat{F}$ is determined canonically only in that same sense.

## 4 Worked examples and sanity checks.

We record a small collection of explicit computations which serve two purposes. First, they confirm that the formal constructions of finite (co)limits in $\widehat{\mathrm{FR}}_k$ behave as one expects from the pointed-set intuition. Second, they illustrate concretely how adjoining a basepoint repairs the most common pathologies of $\mathrm{FR}_k$ (notably, the nonexistence of equalizers coming from empty underlying sets).

**Translations are honest symmetries in the pointed setting.** Let $A \subseteq G$ be a finite nonempty subset and let $g \in G$. The translation $\tau_g : A \to A+g$, $\tau_g(a) = a+g$, is always a Freiman $k$-isomorphism in $\mathrm{FR}_k$ (with inverse $\tau_{-g}$), and hence induces a basepoint-preserving isomorphism

$$\tau_g^+ : (A,G)^+ \longrightarrow (A+g,G)^+, \qquad \tau_g^+(a,0) = (a+g,0), \quad \tau_g^+(\star) = \star.$$

The inverse is $\tau^+_{-g}$, and the identities $\tau^+_g \circ \tau^+_{-g} = \mathrm{id}$ and $\tau^+_{-g} \circ \tau^+_g = \mathrm{id}$ hold on the nose.

This example is a useful reminder of the conceptual difference between $\widehat{\mathrm{FR}}_k$ and $\mathrm{FR}^0_k$. In the normalized category $\mathrm{FR}^0_k$, a translation $\tau_g : (A, G) \to (A + g, G)$ is a morphism only when $g = 0$, since morphisms must preserve 0. Thus $\mathrm{FR}^0_k$ rigidifies additive sets by pinning the identity element, whereas $\widehat{\mathrm{FR}}_k$ retains all translation symmetries by allowing the basepoint to be an arbitrary element. In practice, this means that whenever a construction in $\widehat{\mathrm{FR}}_k$ is easiest to describe after translating the basepoint to 0, we may do so without loss of information and then translate back.

**Empty equalizers in $\mathrm{FR}_k$ and their repair in $\widehat{\mathrm{FR}}_k$.** In $\mathrm{FR}_k$, equalizers need not exist because the set-theoretic equalizer of two maps can be empty, and our objects are required to be nonempty. The simplest instance already occurs for two constant maps with distinct values. Take

$$A := \{0, 1\} \subseteq \mathbb{Z}, \qquad B := \{0, 1\} \subseteq \mathbb{Z},$$

and define Freiman $k$-homomorphisms $f, g : (A, \mathbb{Z}) \to (B, \mathbb{Z})$ by

$$f(0) = f(1) = 0, \qquad g(0) = g(1) = 1.$$

For $k \geq 2$ both maps are Freiman: any additive relation in $A$ is sent to an additive relation in $B$ because each map has constant image. However, the set-theoretic equalizer $\{a \in A : f(a) = g(a)\}$ is empty, and there is therefore no equalizer object in $\mathrm{FR}_k$.

Passing to the pointed completion resolves this immediately. Consider the induced maps

$$f^+, g^+ : (A, \mathbb{Z})^+ \rightrightarrows (B, \mathbb{Z})^+.$$

By definition $f^+(\star) = g^+(\star) = \star$, so the equalizer in $\widehat{\mathrm{FR}}_k$ is never empty: it contains at least the basepoint. In the present example, one checks that

$$\mathrm{Eq}(f^+, g^+) \cong (\{\star\},\ \mathbb{Z} \oplus \mathbb{Z},\ \star),$$

since $f^+(a, 0) \neq g^+(a, 0)$ for $a \in \{0, 1\}$, while $f^+(\star) = g^+(\star)$. Thus the equalizer exists and is the singleton based object, i.e. the zero object of $\widehat{\mathrm{FR}}_k$.

A closely related (and more typical) pointed example is the following. Let

$$(A, G, a_0) := (\{0, 1\}, \mathbb{Z}, 0), \qquad (B, H, b_0) := (\{0, 1\}, \mathbb{Z}, 0),$$

and define basepoint-preserving Freiman $k$-homomorphisms $f, g : (A, G, a_0) \to (B, H, b_0)$ by

$$f(0) = 0,\ f(1) = 0, \qquad g(0) = 0,\ g(1) = 1.$$

Then the equalizer in $\widehat{\mathrm{FR}}_k$ is precisely the singleton based subobject

$$\mathrm{Eq}(f, g) \cong (\{0\}, \mathbb{Z}, 0),$$

since $f$ and $g$ coincide exactly at the basepoint. This is the pointed analogue of the familiar fact from based topology and based sets that equalizers may collapse to the basepoint rather than disappearing.

**An explicit pushout: the wedge/coproduct of based additive sets.** A basic pushout that appears repeatedly is the amalgamation along a singleton basepoint. Let $(A, G, a_0)$ and $(B, H, b_0)$ be objects of $\widehat{\mathrm{FR}}_k$, and consider the span from the singleton based object into each:

$$(\{*\}, 0, *) \longrightarrow (A, G, a_0), \qquad (\{*\}, 0, *) \longrightarrow (B, H, b_0),$$

where each map sends $*$ to the relevant basepoint. (Here we suppress the ambient group of the singleton, since any choice yields an isomorphic singleton based object.) The pushout of this span is, by definition, the coproduct $(A, G, a_0) \amalg (B, H, b_0)$ in $\widehat{\mathrm{FR}}_k$.

Concretely, it is convenient to present this coproduct in an ambient direct sum group with basepoint at the origin. Define a based subset of $G \oplus H$ by

$$A \vee B := \big((A - a_0) \times \{0\}\big) \cup \big(\{0\} \times (B - b_0)\big) \subseteq G \oplus H, \qquad \text{basepoint } (0, 0).$$

The canonical maps

$$\iota_A : (A, G, a_0) \to (A \vee B,\ G \oplus H,\ (0, 0)), \quad \iota_A(a) = (a - a_0, 0),$$

$$\iota_B : (B, H, b_0) \to (A \vee B,\ G \oplus H,\ (0, 0)), \quad \iota_B(b) = (0, b - b_0),$$

are Freiman $k$-homomorphisms and preserve basepoints. Moreover, given any based object $(C, K, c_0)$ and morphisms $\phi : (A, G, a_0) \to (C, K, c_0)$ and $\psi : (B, H, b_0) \to (C, K, c_0)$, there is a unique morphism $\theta : (A \vee B, G \oplus H, (0, 0)) \to (C, K, c_0)$ with $\theta \circ \iota_A = \phi$ and $\theta \circ \iota_B = \psi$, because $\theta$ is forced on the two distinguished summands and these summands intersect only in the basepoint. This realizes $(A \vee B, G \oplus H, (0, 0))$ as the pushout, and hence as the coproduct, of $(A, G, a_0)$ and $(B, H, b_0)$.

For a concrete instance, take $A = \{0, 1\} \subseteq \mathbb{Z}$ with basepoint 0, and $B = \{0, 2\} \subseteq \mathbb{Z}$ with basepoint 0. Then

$$A \vee B = \{(0, 0), (1, 0), (0, 2)\} \subseteq \mathbb{Z} \oplus \mathbb{Z},$$

with basepoint $(0, 0)$, and the coproduct injections are $1 \mapsto (1, 0)$ and $2 \mapsto (0, 2)$.

**An explicit pullback: fiber products as equal-sum constraints.** Pullbacks in $\widehat{\mathrm{FR}}_k$ admit an equally concrete description in elementary examples. Let

$$(A, G, a_0) := (\{0, 1, 2\}, \mathbb{Z}, 0), \qquad (C, K, c_0) := (\{0, 2\}, \mathbb{Z}, 0),$$
$$(B, H, b_0) := (\{0, 1\}, \mathbb{Z}, 0),$$

and define basepoint-preserving maps $f : (A, G, a_0) \to (B, H, b_0)$ and $g : (C, K, c_0) \to (B, H, b_0)$ by

$$f(0) = 0, \ f(1) = 1, \ f(2) = 0, \qquad g(0) = 0, \ g(2) = 0.$$

(Again, for $k \geq 2$ these are Freiman $k$-homomorphisms by a direct inspection of the finitely many additive relations in the source sets.) The pullback $P := A \times_B C$ has underlying set

$$P = \{(a, c) \in A \times C : \ f(a) = g(c)\}.$$

Since $g$ is constant 0, this constraint reduces to $f(a) = 0$, so $a \in \{0, 2\}$. Thus

$$P = \{(0, 0), (2, 0), (0, 2), (2, 2)\} \subseteq \mathbb{Z} \oplus \mathbb{Z}, \qquad \text{basepoint } (0, 0),$$

with the evident projections to $A$ and $C$. In particular, even when the pullback condition cuts down the underlying set substantially, the basepoint survives automatically, ensuring nonemptiness.

These examples capture the general pattern: once basepoints are built into the objects and preserved by morphisms, the universal constructions that would otherwise produce empty sets are forced instead to land on the singleton based object. From the categorical perspective, this is precisely the mechanism by which $\widehat{\mathrm{FR}}_k$ becomes finitely complete and finitely cocomplete while remaining close to the combinatorial content of Freiman homomorphisms.

# 5 Outlook: spans and cospans in $\widehat{\mathrm{FR}}_k$, quantitative decorations, and universal ambient groups.

The availability of finite limits and finite colimits in $\widehat{\mathrm{FR}}_k$ suggests that, beyond ordinary morphisms, it is natural to work with correspondences. Concretely, we may form the bicategory of spans $\mathrm{Span}(\widehat{\mathrm{FR}}_k)$ whose 1-morphisms from $(A, G, a_0)$ to $(B, H, b_0)$ are diagrams

$$(A, G, a_0) \xleftarrow{p} (S, L, s_0) \xrightarrow{q} (B, H, b_0)$$

in $\widehat{\mathrm{FR}}_k$, with composition defined by pullback:

$$(S, p, q) \circ (T, r, u) \ := \ \big(S \times_{(B,H,b_0)} T, \ \pi_S, \ \pi_T\big).$$

Since pullbacks exist and are nonempty (indeed contain the basepoint), the usual coherence data for span composition is available exactly as in any finitely complete category. Dually, we may form the bicategory of cospans $\mathrm{Cospan}(\widehat{\mathrm{FR}}_k)$ using pushouts. In practice, spans encode "multi-valued Freiman maps" and cospans encode amalgamations (gluing along a common based subobject). The pointed setting is particularly well-suited for such constructions because the forced survival of the basepoint prevents degeneracies that would otherwise require ad hoc exclusions.

From the combinatorial viewpoint, spans provide a convenient language for many ubiquitous operations in additive combinatorics. For example, if $X \subseteq A \times B$ is a graph-like subset encoding a relation between $A$ and $B$ (say, a partial Freiman isomorphism, or a Freiman homomorphism defined on a large subset), then $X$ can often be promoted to a span

$$(A, G, a_0) \leftarrow (X,\ G \oplus H,\ (a_0, b_0)) \rightarrow (B, H, b_0),$$

after embedding $X$ into an ambient direct sum group and choosing the evident basepoint. Composition of such relations is then governed by pullback, which corresponds to the set-theoretic fiber product of relations. In this manner, one can model the formal calculus of "changing coordinates" and "passing to a common refinement" without leaving $\widehat{\mathrm{FR}}_k$.

Cospans, on the other hand, organize gluing constructions that resemble wedge sums and identifications along a shared based subobject. If $(C, K, c_0)$ is a common based subobject of $(A, G, a_0)$ and $(B, H, b_0)$, then the pushout

$$(A, G, a_0) \amalg_{(C,K,c_0)} (B, H, b_0)$$

is the canonical recipient of compatible maps out of $A$ and $B$ that agree on $C$. This is a useful formalization of operations such as adjoining auxiliary elements, merging two approximate models along a shared "core", or building larger configurations from smaller ones while controlling additive relations. The fact that pushouts exist for arbitrary based morphisms in $\widehat{\mathrm{FR}}_k$ makes it plausible to study inductive constructions (in particular, those driven by repeated amalgamation) purely categorically.

A further prospect is to impose a quantitative enrichment on $\widehat{\mathrm{FR}}_k$ by decorating objects and/or morphisms with additive-combinatorial parameters. Typical invariants of a finite set $A$ include the doubling constant

$$\sigma(A) \ := \ \frac{|A + A|}{|A|},$$

and the (unnormalized) additive energy

$$E(A) \ := \ \big|\{(a_1, a_2, a_3, a_4) \in A^4 :\ a_1 + a_2 = a_3 + a_4\}\big|.$$

While these are not functorial in the strict sense on $\widehat{\mathrm{FR}}_k$ (e.g. $\sigma$ can increase under non-injective maps because the denominator shrinks), they behave functorially after one introduces appropriate "distortion data". For instance, a Freiman 2-homomorphism $f : A \to B$ canonically induces a well-defined map

$$\widetilde{f} : A + A \longrightarrow B + B, \qquad \widetilde{f}(a_1 + a_2) = f(a_1) + f(a_2),$$

and hence a surjection $A + A \twoheadrightarrow f(A) + f(A)$; thus $|f(A) + f(A)| \le |A + A|$ always holds. This inequality suggests studying morphisms together with quantitative control of fibers of $f$ and of $\widetilde{f}$, rather than attempting to make $\sigma$ strictly monotone.

One reasonable way to formalize such control is to pass from an ordinary category to a locally ordered or weighted category. For example, one may attach to each morphism $f : (A, G, a_0) \to (B, H, b_0)$ numerical parameters such as

$$\kappa_1(f) \ := \ \max_{b \in f(A)} |f^{-1}(b)|, \qquad \kappa_2(f) \ := \ \max_{t \in f(A)+f(A)} \big|\widetilde{f}^{-1}(t)\big|,$$

and then record inequalities of the form

$$|f(A)| \ge \frac{|A|}{\kappa_1(f)}, \qquad |f(A) + f(A)| \ge \frac{|A + A|}{\kappa_2(f)}.$$

Such data can be propagated along composition (with multiplicative bounds), yielding a quantitative calculus in which qualitative categorical constructions (pullbacks, pushouts, (co)equalizers) can be accompanied by explicit bookkeeping of combinatorial losses. In applications, one often tolerates bounded losses (e.g. polylogarithmic or polynomial in $\sigma(A)$); a weighted formalism is a natural place to encode this systematically.

Spans and cospans are also well-adapted to quantitative refinements. A span $(A \leftarrow S \to B)$ can be regarded as a mechanism for comparing $A$ and $B$ through a common refinement $S$, and numerical invariants on $S$ (doubling, energy, or higher Gowers-type counts) can serve as witnesses of the strength of the comparison. For example, if $S$ maps to both $A$ and $B$ with controlled fiber sizes, then estimates for $\sigma(S)$ can be transferred to estimates for $\sigma(A)$ and $\sigma(B)$. Similarly, a cospan $(A \to P \leftarrow B)$ can be used to construct a joint model $P$ for $A$ and $B$; quantitative control of how $A$ and $B$ sit inside $P$ often underlies "modeling lemmas" in additive combinatorics. The categorical language does not produce such bounds by itself, but it provides a fixed scaffold on which quantitative hypotheses and conclusions can be stated uniformly.

A final direction concerns the interaction of $\widehat{\mathrm{FR}}_k$ with universal ambient group constructions. In Freiman theory one associates to an additive set $A$ a "universal" abelian group $\mathcal{U}_k(A)$ generated by (a copy of) $A$ modulo the relations expressing all $k$-term additive relations holding in $A$. The defining property is that any Freiman $k$-homomorphism $f : A \to H$ into an abelian group $H$ factors uniquely through a group homomorphism $\mathcal{U}_k(A) \to H$. In the pointed setting, it is natural to incorporate the basepoint by imposing that it maps to the identity, i.e. to form $\mathcal{U}_k(A, a_0)$ from the translated set $A - a_0$ so that 0 is distinguished. Using the reflector $R(A, G, a_0) = (A - a_0, G)$, one can reduce many questions to the normalized situation and then translate back, and one expects the resulting universal group assignment to be compatible (up to canonical isomorphism) with this normalization/denormalization procedure.

If such a universal ambient group functor is made explicit at the level of $\widehat{\mathrm{FR}}_k$, then spans in $\widehat{\mathrm{FR}}_k$ can be transported to spans of abelian groups, where additional algebraic tools become available (kernels, cokernels, ranks, torsion, and homological invariants). Conversely, algebraic constructions on universal groups can sometimes be pulled back to $\widehat{\mathrm{FR}}_k$ as invariants of based additive sets. A basic test case is whether pushouts in $\widehat{\mathrm{FR}}_k$ map to pushouts (amalgamated sums) of universal groups, or at least admit comparison morphisms that measure the failure of exact preservation. Understanding precisely how universal group adjunctions interact with finite (co)limits in $\widehat{\mathrm{FR}}_k$ appears to be a promising route toward a conceptual explanation of why many "approximate group" arguments can be organized around a small set of universal constructions.